

Graviton production from extra dimensions

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Abstract

Graviton production due to collapsing extra dimensions is studied. The momenta lying in the extra dimensions are taken into account. A D -dimensional background is matched to an effectively four-dimensional standard radiation dominated universe. Using observational constraints on the present gravitational wave spectrum, a bound on the maximal temperature at the beginning of the radiation era is derived. This expression depends on the number of extra dimensions, as well as on the D -dimensional Planck mass. Furthermore, it is found that the extra dimensions have to be large.

1 Introduction

Our observable universe is four-dimensional. However fundamental theories, like for example string theories, require for consistency more than four space-time dimensions. In the standard Kaluza-Klein approach [1] the extra dimensions are compact and curled up to very small size, roughly of the order of the Planck scale, $\ell_P \sim 10^{-33}\text{cm}$, implying that space-time is effectively four-dimensional. However recently, with the discovery of brane solutions to string/M-theory, the idea of large extra dimensions has attracted much attention [2]. In this last approach, it is proposed that ordinary

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matter is confined to a three-dimensional sub-manifold, a three-brane, which is embedded in a higher dimensional space-time; the graviton on the other hand is allowed to propagate freely through the whole space-time. Neglecting the brane tension, i.e. the energy density per unit three-volume of the brane, and considering compact dimensions, one re-introduces the Kaluza-Klein picture. However, in this case the extra dimensions do not have to be small. Newtonian gravity has been tested and found to hold down to scales of order of 1 mm [3]. Below this scale, gravity could in principle be higher than four-dimensional. In higher dimensional gravity, the four-dimensional Planck scale is no longer a fundamental scale; the higher dimensional Planck scale, M_D , becomes instead the fundamental scale. This allows to explain the huge difference between the electroweak and the four-dimensional Planck scale, known as the hierarchy problem. Assuming for simplicity that the n extra dimensions form an n -torus which has the same radius R in each direction, and using Gauss' law, the D -dimensional and the four-dimensional Planck masses, M_4 and M_D respectively, are related by [4] [5]

$$M_4^2 = R^n M_D^{n+2} , \quad (1.1)$$

where R^n is the volume of the n extra dimensions. Thus, taking M_D to be of the order of the electroweak scale, $M_D \sim 1\text{TeV}$, the huge difference between M_4 and the electroweak scale can be explained as due to the large size of the extra dimensions.

Here we are going to study the effects of time-varying extra dimensions on the graviton production. We assume that the background space-time can be written as a direct product of an internal space, i.e. the extra dimensions, and a four-dimensional external space-time. Furthermore, during an initial phase, the internal space is contracting and the four-dimensional external space-time is expanding. At some time this is matched to a radiation dominated flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe with the internal dimensions frozen at constant size. Due to the changing background metric, there will be particle production. Here we will be in particular concerned with the production of gravitons. Using observational bounds on graviton spectra it is possible to derive a relation between the D -dimensional Planck mass and the temperature at the time of transition. Usually it is assumed that there are no massive modes excited, i.e. momenta lying in the internal space are not taken into account. Here however, we are going to assume that the internal momenta are excited and we are interested in their effect on the spectral energy density in four dimensions. In order to determine this effect, the internal momenta are integrated out. This will lead to a final expression for the spectral energy density, which will be just depending on the four-dimensional momenta.

The formalism to deal with metric perturbations has already been developed in Ref. [6]. Particle production in higher dimensional space-times has been also the subject of Refs. [7] and [8].

2 Graviton production

Consider a $D(=d+n+1)$ -dimensional space-time with line element,

$$ds^2 = a^2(\eta) [d\eta^2 - \delta_{ij} dx^i dx^j] - b^2(\eta) \delta_{AB} dy^A dy^B , \quad (2.2)$$

where $d=3$, and the indices $i, j = 1, \dots, 3$ and $A, B = 4, \dots, 3+n$. Thus, a and b denote the scale factors for the internal and external spaces respectively, η stands for the conformal time, and δ_{ij} is the Kronecker symbol. Assume that the extra dimensions are compact and, for simplicity, that

they all have the same size, i.e. $0 \leq y^A \leq 2\pi r$. Then the length-scale, R , characterizing the physical size of the extra dimensions is $R = 2\pi r b$.

The higher dimensional phase is matched to a radiation dominated universe. It is assumed that the extra dimensions are dynamical during the first stage. At the onset of the radiation dominated era they become static.

Hence, the evolution of the background will be parametrized as follows,

$$\begin{aligned} a(\eta) &= a_1 \left(-\frac{\eta}{\eta_1} \right)^\sigma, & b(\eta) &= b_1 \left(-\frac{\eta}{\eta_1} \right)^\lambda, & \text{for } \eta < -\eta_1 \\ a(\eta) &= a_1 \left(\frac{\eta + 2\eta_1}{\eta_1} \right), & b(\eta) &= b_1, & \text{for } \eta \geq -\eta_1. \end{aligned} \quad (2.3)$$

In the following we set $a_1 = b_1 = 1$. In Figure 1 we present the evolution of the background for $n = 8$ extra dimensions.

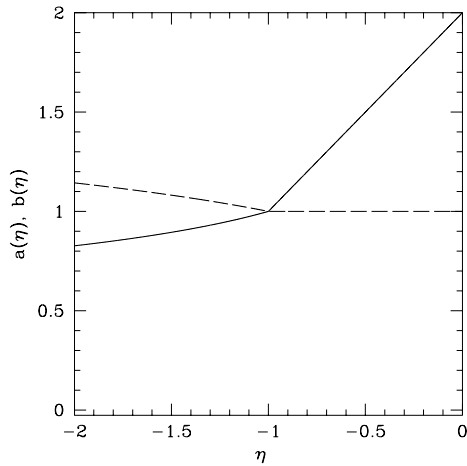


Figure 1: The evolution of the background during a twelve-dimensional and an effectively four-dimensional stage. Here the transition takes place at $\eta = -\eta_1 = -1$, and the number of extra dimensions is given by $n = 8$. The solid line shows the scale factor $a(\eta)$ of the four-dimensional space-time and the dashed line shows the scale factor $b(\eta)$ of the eight-dimensional extra space.

Within the context of the Kaluza-Klein theory, there are expected oscillations of the b -field about the minimum. This is not considered here, since this issue represents a question on its own. As it can be seen from Eq. (2.3), we consider a power-law behavior for the b -field. Parametric resonance in the context of a higher D -dimensional Kaluza-Klein theory has been only studied, to our knowledge, for Kaluza-Klein modes [9].

The tensor perturbations of the D -dimensional metric $G_{\tilde{A}\tilde{B}}$ take the form [6]

$$\delta G_{\tilde{A}\tilde{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a^2 \gamma_{ij}(x, y) & 0 \\ 0 & 0 & -b^2 \gamma_{AB}(x, y) \end{pmatrix}.$$

We consider that the unperturbed D -dimensional metric is a direct product, thus there are no cross-terms in the above expression for the tensor perturbations. Only γ_{ij} describes gravitational waves

in the four-dimensional space-time. Although γ_{AB} describes gravitational waves in the n extra dimensions, it is interpreted as a scalar (matter) perturbation in the four-dimensional space-time of the observer. Both types of gauge-invariant tensor modes satisfy the D -dimensional Klein-Gordon equation for a massless scalar field. Here, we are only interested in the modes which behave as gravitational waves in the four-dimensional space-time. Writing $\gamma_{ij} = \phi(t, x, y)e_{ij}$, where e_{ij} is the polarization tensor, the amplitude of the gravitational waves ϕ satisfies the same equation as a minimally coupled scalar field, i.e. [6] [10]

$$\square\phi = 0 ,$$

where \square denotes the d'Alembert operator in D -dimensions. In the above background, in Fourier space, this yields to the mode equation for the canonical field $\Phi = ab^{n/2}\phi$,

$$\Phi_l'' + \left[k^2 + \left(-\frac{\eta}{\eta_1} \right)^{2\beta} q^2 - \frac{N}{\eta^2} \right] \Phi_l = 0 , \quad (2.4)$$

where the three-vector \mathbf{k} lies in the external space, the n-vector \mathbf{q} is in the internal (compact) space, l stands for (\mathbf{k}, \mathbf{q}) , and

$$\beta \equiv \sigma - \lambda$$

$$N \equiv \frac{(ab^{n/2})''\eta^2}{ab^{n/2}} = \sigma(\sigma - 1) + n\sigma\lambda + n(n - 2)\lambda^2/4 + n\lambda(\lambda - 1)/2 .$$

Furthermore, the modulus of the external and internal momentum vectors is denoted by $k \equiv \sqrt{|\mathbf{k}|}$ and $q \equiv \sqrt{|\mathbf{q}|}$, respectively. Using the compactness of the extra dimensions the components q^A can be written as $q^A = \mathcal{N}_A/r$, where \mathcal{N}_A are integers. This leads for the modulus of \mathbf{q} to

$$q = \frac{\mathcal{N}}{r} , \quad (2.5)$$

with \mathcal{N} given by $\mathcal{N} = \sqrt{\sum \mathcal{N}_A^2}$. For $\beta = -1$ exact solutions can be found, which are discussed in Ref. [8]. However, for general values of β there is no known exact solution. Therefore, we follow Ref. [11] to find an approximate solution. In the following, it is assumed that $\sigma < 0$ and $\lambda > 0$, which implies $\beta < 0$. This describes a universe with a collapsing internal (extra) dimensions and expanding external dimensions, before the transition to an effectively four-dimensional space-time.

2.1 Mode evolution in the presence of dynamical internal dimensions

During the higher dimensional state the background is given by the vacuum Kasner metric. This constrains the exponents λ and σ , and thus β , and allows to write the metric as a function of the number of extra dimensions n .

Namely, the Kasner conditions $d\alpha_E + n\alpha_I = 1$ and $d\alpha_E^2 + n\alpha_I^2 = 1$ imply for $d = 3$ and $n \neq 0$, that

$$\alpha_E = \frac{1}{3} \left[1 - \frac{n}{3+n} \pm (-n) \sqrt{\frac{3}{n} \frac{2+n}{(3+n)^2}} \right] \quad (2.6)$$

$$\alpha_I = \frac{1}{3+n} \pm \sqrt{\frac{3}{n} \frac{2+n}{(3+n)^2}} . \quad (2.7)$$

These are related to the exponents σ and λ by

$$\sigma = \frac{\alpha_E}{1 - \alpha_E} \quad (2.8)$$

$$\lambda = \frac{\alpha_I}{1 - \alpha_E} . \quad (2.9)$$

Hence, calculating the parameter N in Eq. (2.4) yields $N = -(1/4)$, independently of the number of extra dimensions. This can be understood by realizing that $ab^{n/2} = (a^3b^n/a)^{1/2} = (\text{Volume}/a)^{1/2}$, where the volume is constrained by the first Kasner condition.

The parameter β , which describes the difference in the expansion rates of the internal and external space, takes values $-1 \leq \beta < -1/(1 + \sqrt{3})$, where the lower limit corresponds to $n = 1$ and the upper one to the limit of a large number of extra dimensions n .

The approximate solution of the mode equation, Eq. (2.4), is found by solving it in two different regimes. Following the same approximation as in Ref. [11], two cases emerge.

Case (i): Assume that in Eq. (2.4) the square of the three-momentum k always dominates over the term involving the internal momentum q . For very early times, $\eta \rightarrow -\infty$, or equivalently very large k , hence very short wavelengths, k^2 also dominates over the η^{-2} -term. And thus, the solutions are (in-coming) plane waves. However, for times closer to the transition at $\eta = -\eta_1$, or for smaller k , hence longer wavelengths, $k^2 < 1/\eta^2$. Hence, this implies for the modulus of the internal momentum q , that $q < (-\eta/\eta_1)^{-\beta}k$, and thus together with $\eta_1 \equiv 1/k_1$, the condition reads: $q/k < (k/k_1)^\beta$. Note that this case also includes the case $q = 0$. The in-coming state is effectively four-dimensional. Then Eq. (2.4) reduces to a Bessel equation which is adequately, i.e. correctly normalized to an in-coming vacuum state, solved by

$$\Phi_l = \sqrt{\frac{|k\eta|}{k}} H_\mu^{(2)}(|k\eta|) \quad \text{for} \quad \eta < -\eta_1 , \quad (2.10)$$

with $H_\mu^{(2)}$ the Hankel function of the second kind and $\mu^2 \equiv (1/4) + N$. Together with $N = -1/4$ the index of the Hankel function is identically vanishing, $\mu = 0$. This solution is matched on super-horizon scales, i.e. $k\eta_1 \ll 1$, to a radiation dominated universe at $\eta = -\eta_1$. Thus, the canonical field Φ_l and its first derivative are given respectively by

$$\begin{aligned} \Phi_l|_{\eta=-\eta_1} &\sim -i\frac{2}{\pi}\sqrt{\eta_1}\ln(k\eta_1) \\ \Phi'_l|_{\eta=-\eta_1} &\sim -\frac{i}{\pi}\eta_1^{-\frac{1}{2}}[2 + \ln(k\eta_1)] . \end{aligned} \quad (2.11)$$

Case (ii): Assume that the \mathbf{q} -term becomes dominant before the perturbation in D dimensions becomes super-horizon, hence $k^2 + q^2(-\eta/\eta_1)^{2\beta} > 1/\eta^2$. Thus, the mode equation can be approximated by

$$\Phi_l'' + \left[k^2 + q^2 \left(-\frac{\eta}{\eta_1} \right)^{2\beta} \right] \Phi_l = 0, \quad (2.12)$$

which is approximately solved by (cf [11])

$$\Phi_l \simeq \frac{\exp \left[i\eta \sqrt{k^2 + \kappa^2 q^2} \left(-\frac{\eta}{\eta_1} \right)^{2\beta} \right]}{\left(\frac{\pi}{2} \right)^{\frac{1}{2}} \left[k^2 + q^2 \left(-\frac{\eta}{\eta_1} \right)^{2\beta} \right]^{\frac{1}{4}}}, \quad (2.13)$$

where $\kappa \equiv 1/(1+\beta)$, $\beta \neq -1$. The case $\beta = -1$ can be solved exactly and it is discussed in Ref. [8].

At conformal time $\eta = \eta_q$ the internal momentum becomes dominant over the external momentum and finally the η^{-2} -term dominates. For $\eta > \eta_q$ the mode equation can be approximated by

$$\Phi_l'' + \left[q^2 \left(-\frac{\eta}{\eta_1} \right)^{2\beta} - \frac{N}{\eta^2} \right] \Phi_l = 0, \quad (2.14)$$

which is solved by [11]

$$\Phi_l = c_q^{(1)} \sqrt{|q\eta|} H_{\mu\kappa}^{(1)} \left(|q\eta| \kappa \left(-\frac{\eta}{\eta_1} \right)^\beta \right) - i c_q^{(2)} \sqrt{|q\eta|} H_{\mu\kappa}^{(2)} \left(|q\eta| \kappa \left(-\frac{\eta}{\eta_1} \right)^\beta \right). \quad (2.15)$$

Matching Eq. (2.15) to Eq. (2.13) leads to $c_q^{(1)} = 0$ and $c_q^{(2)} = i/\sqrt{q}$. Hence, for $|q\eta_1| \ll 1$, i.e. super-horizon perturbations, and using $H_0^{(2)}(z) \sim -i(2/\pi) \ln z$, one obtains that Φ_l and its first derivative at $\eta = -\eta_1$ are given respectively by

$$\begin{aligned} \Phi_l|_{\eta=-\eta_1} &\sim -i \frac{2}{\pi} \eta_1^{\frac{1}{2}} \ln(\kappa q \eta_1) \\ \Phi_l'|_{\eta=-\eta_1} &\sim -\frac{i}{\pi} \eta_1^{-\frac{1}{2}} \ln(\kappa q \eta_1) - i \frac{2}{\pi} (\beta + 1) \eta_1^{-\frac{1}{2}}. \end{aligned} \quad (2.16)$$

2.2 Mode evolution with static internal dimensions

The higher dimensional state evolves instantaneously into an effectively four-dimensional radiation dominated universe at $\eta = -\eta_1$. The extra dimensions stay at a constant size after the transition, with scale factors given by Eq. (2.3) for $\eta > -\eta_1$. Then the mode equation, Eq. (2.4), takes the form

$$\Phi_l'' + \left[k^2 + \left(\frac{\eta + 2\eta_1}{\eta_1} \right)^2 q^2 \right] \Phi_l = 0. \quad (2.17)$$

This can be transformed into the equation for parabolic cylinder functions by introducing $z \equiv (2q/\eta_1)^{1/2}(\eta + 2\eta_1)$ and $a \equiv -(\eta_1/2q)k^2$ [8]. This yields

$$\frac{d^2 \Phi_l}{dz^2} + \left(\frac{z^2}{4} - a \right) \Phi_l = 0. \quad (2.18)$$

Its complex standard solution is given by [12]

$$\Phi_l(z) = \mathcal{B} [c_- E(a, z) + c_+ E^*(a, z)] , \quad (2.19)$$

where for later convenience the constants are chosen to be the Bogoliubov coefficients c_{\pm} and \mathcal{B} is a normalization factor. With the Wronskian condition on the mode functions (see for example Ref. [7]), i.e. $\Phi_l'^* \Phi_l - \Phi_l' \Phi_l^* = i$, and the normalization of the Bogoliubov coefficients $|c_+|^2 - |c_-|^2 = 1$, together with the expression for the Wronskian of $E(a, z)$, $E^*(a, z)$ [12], the normalization factor \mathcal{B} is found to be $\mathcal{B} = [\eta_1/(2q)]^{1/4}/\sqrt{2}$.

Hence the mode solution is given by

$$\Phi(z) = \frac{1}{\sqrt{2}} \left(\frac{\eta_1}{2q} \right)^{\frac{1}{4}} [c_- E(a, z) + c_+ E^*(a, z)]. \quad (2.20)$$

In order to make progress the solutions will be considered in two regimes of approximation which will comprise cases (iii) and (iv).

Case (iii): Assume that $|a| > z^2/4 \Leftrightarrow k^2 > [(\eta + 2\eta_1)/\eta_1]^2 q^2$. Using the limiting behaviour given in Ref. [12] ([19.22.4]), E can be approximated in this limit as

$$E(a, \eta) \sim \sqrt{2} \frac{1}{2^{3/4}} \left| \frac{\Gamma(\frac{1}{4} + \frac{1}{2}ia)}{\Gamma(\frac{3}{4} + \frac{1}{2}ia)} \right|^{\frac{1}{2}} \exp \left[-\frac{1}{4} \left(\frac{q}{k} \right)^2 \left(\frac{\eta + 2\eta_1}{\eta_1} \right)^2 \right] e^{ik(\eta + 2\eta_1)} \quad (2.21)$$

and a similar expression for the complex conjugate E^* . Using

$$\left| \frac{\Gamma(\frac{1}{4} + \frac{1}{2}ia)}{\Gamma(\frac{3}{4} + \frac{1}{2}ia)} \right|^{\frac{1}{2}} \sim \left| \frac{a}{2} \right|^{-\frac{1}{4}},$$

the mode functions in this limit evolve according to

$$\Phi_l \sim \frac{1}{\sqrt{2k}} \exp \left[-\frac{1}{4} \left(\frac{q}{k} \right)^2 \left(\frac{\eta + 2\eta_1}{\eta_1} \right)^2 \right] [c_- e^{ik(\eta + 2\eta_1)} + c_+ e^{-ik(\eta + 2\eta_1)}]. \quad (2.22)$$

Case (iv): Assume that $|a| < z^2/4 \Leftrightarrow k^2 < [(\eta + 2\eta_1)/\eta_1]^2/q^2$. With the approximation of $E(a, z)$ [12] in this limit the evolution of the modes is given by

$$\Phi_l(\eta) \sim \sqrt{\frac{\eta_1}{2q(\eta + 2\eta_1)}} \left[c_- \exp \left(i \frac{q}{2\eta_1} (\eta + 2\eta_1)^2 \right) + c_+ \exp \left(-i \frac{q}{2\eta_1} (\eta + 2\eta_1)^2 \right) \right]. \quad (2.23)$$

2.3 Graviton spectra

Gravitons are produced due to the changing background space-time. Their spectral energy density might allow conclusions about the early history of the universe. The Bogoliubov coefficients transform the in-state into the out-state and determine the number of created particles. As it turns out there are three possible combinations of the four mode solutions discussed previously. It is found that the evolution of modes from the regime of validity of cases (i) and (iii), (i) and (iv), as well as (ii) and (iv) is possible. For these three regimes, the Bogoliubov coefficients have to be calculated. Then, the spectral energy density is found by simply adding up the contributions from the three parts. In order to find the total spectral energy density in the effectively four-dimensional

space-time after the transition, the momenta lying in the internal (extra) space are integrated out. In this way the contribution of energy due to the modes in the extra space can be estimated.

The Bogoliubov coefficients are calculated by matching the mode functions and their first derivatives at the transition time $\eta = -\eta_1$, on super-horizon scales. The resulting expressions together with the appropriate range of validity are given below.

Case (i) matched to (iii):

$$e^{ik\eta_1} c_- \sim -\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{k\eta_1}} e^{\frac{1}{4}\left(\frac{q}{k}\right)^2} \left[2 + \left(1 + \left(\frac{q}{k} \right)^2 + 2ik\eta_1 \right) \ln(k\eta_1) \right]. \quad (2.24)$$

This holds for $\frac{q}{k} < a^{-1}$.

Case (i) matched to (iv):

$$c_- e^{i\frac{q\eta_1}{2}} \sim -\frac{\sqrt{2}}{\pi\sqrt{q\eta_1}} [1 + (1 + iq\eta_1) \ln(k\eta_1)], \quad (2.25)$$

which holds for $a^{-1} < \frac{q}{k} < \left(\frac{k_1}{k}\right)^{-\beta}$.

Case (ii) matched to (iv):

$$c_- e^{i\frac{q}{2}\eta_1} \sim -\frac{\sqrt{2}}{\pi\sqrt{q\eta_1}} [\beta + 1 + (1 + iq\eta_1) \ln(\kappa q\eta_1)]. \quad (2.26)$$

This holds for $\frac{q}{k} > \left(\frac{k_1}{k}\right)^{-\beta}$.

Case (ii) matched to (iii):

Clearly, this possibility is not allowed within our set-up, where the extra (internal) dimensions are collapsing while the external four dimensions are expanding. This implies $\beta < 0$, and therefore case (ii) requires $q/k > 1$, while case (iii) holds for the opposite limit, namely $q/k < 1$. Therefore, our scenario does not allow the matching of cases (ii) and (iii).

In order to compare with observations it is useful to calculate the spectral energy density. The total energy density of created particles is given by

$$\rho = 2 \frac{R^n}{(2\pi)^{n+3}} \int \left[\frac{k^2}{a^2} + \frac{q^2}{b^2} \right]^{\frac{1}{2}} |c_-|^2 dV_{\text{phys}} \quad (2.27)$$

and $dV_{\text{phys}} = dV_{\text{com}}/(a^3 b^n)$, where it is assumed that the comoving volume consists of two spheres, i.e.

$$dV_{\text{com}} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} k^{d-1} dk \wedge \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} q^{n-1} dq,$$

which for $d = 3$ implies

$$\rho = \frac{R^n}{(2\pi)^{n+3}} 16 \frac{\pi^{1+\frac{n}{2}}}{\Gamma(\frac{n}{2})} a^{-4} b^{-n} \int \left[1 + \left(\frac{a}{b} \right)^2 \left(\frac{q}{k} \right)^2 \right]^{\frac{1}{2}} k^3 q^{n-1} |c_-|^2 dk dq. \quad (2.28)$$

Hence the spectral energy density $\rho(k) = d\rho/d\log k$ is given by

$$\rho(k) = \frac{R^n}{(2\pi)^{n+3}} 16 \frac{\pi^{1+\frac{n}{2}}}{\Gamma(\frac{n}{2})} \left(\frac{k}{a} \right)^4 \left(\frac{k}{b} \right)^n \int dY Y^{n-1} \left[1 + \left(\frac{a}{b} \right)^2 Y^2 \right]^{\frac{1}{2}} |c_-|^2, \quad (2.29)$$

where $Y \equiv q/k$.

As explained above there are three contributions to the spectral energy density. The notation $\rho_{(i)-(iii)}$ stands for the contribution resulting from the Bogoliubov coefficients calculated from the matching of cases (i) and (iii), and in a similar way for the other cases. For $\rho_{(i)-(iii)}$, the integral

$$\int_0^{1/a} dY Y^{n-1} \left[1 + \left(\frac{a}{b} \right)^2 Y^2 \right]^{1/2} |c_-|^2$$

has to be calculated. For $\rho_{(i)-(iv)}$, the corresponding one. In order to calculate $\rho_{(ii)-(iv)}$, we introduce an upper cut-off in the integral, i.e.

$$\int_{(k_1/k)^{-\beta}}^{q_{\max}/k} dY Y^{n-1} \left[1 + \left(\frac{a}{b} \right)^2 Y^2 \right]^{1/2} |c_-|^2.$$

At the transition time $\eta = -\eta_1$ the metric is continuous but its first derivative is not. This leads to the so-called sudden transition approximation. It basically means that for the modes with periods much greater than the duration of the transition phase, the transition can be considered as instantaneous. It is known that this type of approximation leads to an ultra-violet divergency [7][13]. This justifies the introduction of an upper cut-off q_{\max} . Neglecting sub-leading terms for the total spectral energy density, $\rho(k) = \rho_{(i)-(iii)} + \rho_{(i)-(iv)} + \rho_{(ii)-(iv)}$, the following expression is found

$$\begin{aligned} \rho(k) &\sim R^n \frac{2^{-n}}{a^{4+n}} \frac{\pi^{-4-\frac{n}{2}}}{n\Gamma(\frac{n}{2})} k_1^{4+n} \left(\frac{k}{k_1} \right)^{3+n} \left(\ln \frac{k}{k_1} \right)^2 \\ &+ R^n \frac{2^{2-n}}{a^3} \frac{\pi^{-4-\frac{n}{2}}}{n\Gamma(\frac{n}{2})} k_1^{4+n} \left(\frac{k}{k_1} \right)^{3+n(\beta+1)} \left(\ln \frac{k}{k_1} \right)^2 \\ &+ R^n \frac{2^{2-n}}{a^3} \frac{\pi^{-4-\frac{n}{2}}}{n\Gamma(\frac{n}{2})} k_1^{4+n} \left(\frac{k}{k_1} \right)^3 \left(\frac{q_{\max}}{k_1} \right)^n \left(\ln \kappa \frac{q_{\max}}{k_1} \right)^2. \end{aligned} \quad (2.30)$$

This can be expressed in terms of the fractional energy density (dimensionless parameter), $\Omega(k) \equiv \rho(k)/\rho_c$, where the critical energy density ρ_c is given in terms of the D -dimensional Planck mass, M_D , as $\rho_c = [3/(8\pi)] R^n M_D^{n+2} H^2$. We thus find that the fractional energy density of produced gravitons $\Omega_{\text{GW}}(\eta)$ at a given time η , with respect to the fraction of critical energy density in radiation, $\Omega_\gamma(\eta)$, defined as $\Omega_\gamma(\eta) = (H_1/H)^2 (a_1/a)^4$, is

$$\Omega_{\text{GW}}(\eta) \sim \mathcal{A}_1 a^{-n} \left(\frac{H_1}{M_D} \right)^{2+n} \Omega_\gamma(\eta) \left(\frac{k}{k_1} \right)^{3+n} \left(\ln \frac{k}{k_1} \right)^2$$

$$\begin{aligned}
& + \mathcal{A}_2 a \left(\frac{H_1}{M_D} \right)^{2+n} \Omega_\gamma(\eta) \left(\frac{k}{k_1} \right)^{3+n(\beta+1)} \left(\ln \frac{k}{k_1} \right)^2 \\
& + \mathcal{A}_2 a \left(\frac{H_1}{M_D} \right)^{2+n} \Omega_\gamma(\eta) \left(\frac{k}{k_1} \right)^3 \left(\frac{q_{max}}{k_1} \right)^n \left(\ln \kappa \frac{q_{max}}{k_1} \right)^2 .
\end{aligned} \tag{2.31}$$

The constants \mathcal{A}_i contain all the numerical factors. Furthermore, it was assumed that the maximal frequency k_1 in the four-dimensional space-time is of the same order as the curvature scale at the time of transition, i.e. $k_1 \sim H_1$.

This expression can be compared with the one for no excited internal momenta. It can be easily recovered from the matching of case (i) to (iii), without carrying out the integration. It is found that

$$\Omega_{\text{GW} \mid q=0}(\eta) = \mathcal{A} \left(\frac{H_1}{M_4} \right)^2 \Omega_\gamma(\eta) \left(\frac{k}{k_1} \right)^3 \left(\ln \frac{k}{k_1} \right)^2 . \tag{2.32}$$

As it can be seen from the expression for the fractional energy density, Eq. (2.31), the effect of the momenta in the extra dimensions is two-fold. On the one hand, there are two contributions which are growing with the scale factor, and therefore might soon dominate the energy density of the universe and overclose it. On the other hand, the cut-off q_{max} has to be chosen in such a way that its contribution does not dominate, i.e. $q_{\text{max}} \sim (\beta + 1)k_1$. Thus, the cut-off of the momenta in the extra space has to be less than the one in the four-dimensional space-time, $q_{\text{max}} < k_1$.

There are several constraints on the gravitational wave spectrum. The strongest ones come from big bang nucleosynthesis and the prevention of recollapse of the universe. Both of these hold for the whole range of frequencies. Big bang nucleosynthesis successfully predicts the production of light elements. However, everything depends crucially on the expansion rate at the time of nucleosynthesis. This in turn is determined by the total energy density at that time. Therefore, any extra contributions at that time are constrained. The total energy density of gravitational waves at the present time has to satisfy [15][16]

$$\int_{f=0}^{f=\infty} d(\log f) h_0^2 \Omega_{\text{GW}}(f) \leq 0.227 \Omega_\gamma , \tag{2.33}$$

where $0.5 < h_0 < 0.85$ parametrizes the experimental uncertainty for the present value of the Hubble parameter H_0 . One might argue that, as a first approximation, $h_0^2 \Omega_{\text{GW}}(f) \leq 0.227 \Omega_\gamma$, which leads to

$$\mathcal{A}_2 a_0 \left(\frac{H_1}{M_D} \right)^{2+n} < 0.227 . \tag{2.34}$$

Furthermore, a_0 is given by

$$a_0 = \Omega_{\gamma,0}^{-1/4} \left(\frac{H_1}{H_0} \right)^{1/2} \sim 5 \times 10^{31} \left(\frac{H_1}{M_4} \right)^{1/2} ,$$

where the subscript 0 denotes the present epoch. Assuming that the standard behaviour of radiation holds up to the transition scale, the Hubble parameter H and the temperature T are related by (see for example [17])

$$H = 1.66 g_*^{\frac{1}{2}} \frac{T^2}{M_4} , \tag{2.35}$$

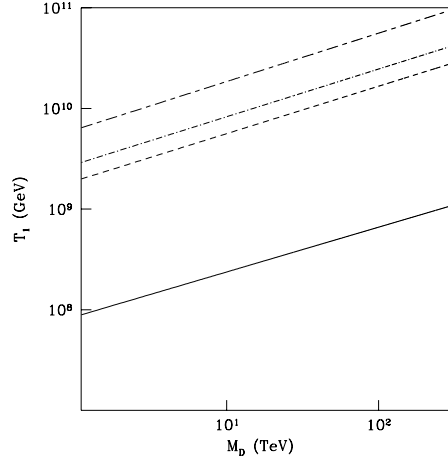


Figure 2: The maximal temperature at the beginning of the radiation dominated FLRW stage as a function of the D -dimensional Planck mass. The solid line shows $n = 2$ extra dimensions, the long dashed one is for $n = 6$, the dashed-dotted line for $n = 7$ and the long/short-dashed one for $n = 10$.

where $g_*(T)$ counts the total number of effectively massless degrees of freedom. For temperatures $T > 300$ GeV, one has $g_* = 106.75$. Using Eq. (2.35) for the value of the Hubble parameter at the time of transition, H_1 , the bound given by Eq. (2.34) can be written in terms of the temperature at the transition, i.e. at the beginning of the radiation dominated FLRW stage. The maximal temperature T_1 is given when the bound is satisfied. It is found to be

$$T_1 = \left[3I \frac{(1.211)^{3+n}}{5} \frac{10^{32+22n}}{[1.66g_*^{1/2}]^{\frac{5}{2}+n} 2^{5-n}} \pi^{3+\frac{n}{2}} n \Gamma\left(\frac{n}{2}\right) \right]^{\frac{1}{5+2n}} \left(\frac{M_D}{1\text{TeV}} \right)^{\frac{2+n}{5+2n}} \text{GeV} , \quad (2.36)$$

where $I = 0.227$ is our bound. This is shown in Figure 2. It should be noted that the range of the D -dimensional Planck mass is constrained by the upper size of the extra dimensions, $R_{\text{max}} \sim 1\text{mm}$. This gives the constraint

$$M_D \geq \left(2 \times 10^{(32/n)-16} \right)^{1/(1+\frac{2}{n})} \text{TeV} . \quad (2.37)$$

Therefore, there is a lower bound on T_1 . The minimal value of M_D and the minimal value of T_1 as a function of the number n of extra dimensions are shown in Figure 3. Assuming that T_1 is the same as the reheating temperature T_{RH} , Eqs. (2.36) and (2.37) imply a lower bound on T_{RH} . More precisely, for one extra dimension ($n = 1$), the lower bound for the D -dimensional Planck mass is $M_D > 10^5 \text{TeV}$, leading to a lower bound for the reheating temperature $T_{\text{RH}} > 3 \times 10^9 \text{GeV}$. In a similar way, for two extra dimensions ($n = 2$), one obtains $M_D > 1 \text{TeV}$ and $T_{\text{RH}} > 10^7 \text{GeV}$. Finally, for $n = 3$ one has $M_D > 10^{-3} \text{TeV}$ and $T_{\text{RH}} > 5 \times 10^6 \text{GeV}$.

Big-bang nucleosynthesis must proceed in the standard way, implying that the reheating temperature must be higher than $\sim 10 \text{MeV}$. This requirement is satisfied in our model, as the above analysis shows. Standard inflationary models, on the other hand, lead to a higher reheating temperature than the lower bounds we found above. However, one must keep in mind that above we

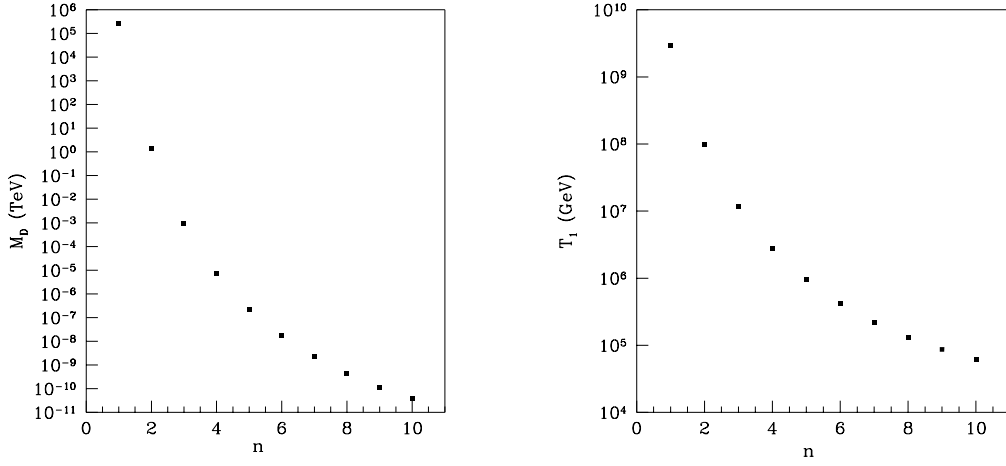


Figure 3: In the left panel the allowed minimal value of the D -dimensional Planck mass M_D is shown as a function of the number of extra dimensions n . In the right panel the minimal value of the temperature at the beginning of the radiation dominated stage, T_1 , is shown as a function of n .

only gave the lower bounds, while the actual reheating temperature can be much higher indeed. In addition, a higher value can be easily achieved increasing the value of the D -dimensional Planck mass, provided it always satisfies the constraint given in Eq. (2.37). Finally, reheating from a higher number of collapsing extra dimensions favours larger values of M_D , and thus smaller extra dimensions.

The other strong constraint comes from the requirement that the additional energy density should not overclose the universe. Thus, the condition $\Omega_{\text{GW}}(\omega) < 1$ has to hold for all frequencies. However, if the big bang nucleosynthesis constraint is satisfied, then indeed $\Omega_{\text{GW}} < 1$.

There are additional constraints coming from large scales as probed by the cosmic microwave background explorer (COBE). Strong perturbations can change the gravitational potential experienced by the microwave background photons. This causes a red-shift of these photons and hence a fluctuation in temperature of the cosmic microwave background [15]. There is, in addition, a bound due to pulsars whose signal is delayed, if a gravitational wave passes through between the earth and the pulsar [15]. However, both these constraints apply only for a range of frequencies and they are in general less tight than the ones mentioned earlier. Furthermore, since the spectrum is increasing with increasing frequency, the most stringent bounds will come from large frequencies, i.e. small wavelengths.

In conclusion, we find that the excited internal modes are compatible with observations, provided the temperature at the transition time to an effectively four-dimensional universe, is less than its maximum, given in Eq. (2.36). Thus, recalling that $H_1 \sim T_{\text{RH}}^2/M_4$, $H_1 \sim k_1$, $q_{\text{max}} < k_1$ and $q \sim R^{-1}$, this implies that the size R of the extra dimensions has to be $R > M_4/T_{\text{RH}}^2$. As we have shown, the lower bound of the reheating temperature depends on the number n of extra dimensions and the value of the D -dimensional Planck mass M_D . With values of M_D , such that the constraint Eq. (2.37) is satisfied, one can easily check that our analysis leads to the requirement that the size of the internal space must be much larger than the four-dimensional Planck length

$$R \gg \ell_{\text{Pl}} , \quad (2.38)$$

where ℓ_{Pl} is the four-dimensional Planck length ($\ell_{\text{Pl}} \sim 10^{-19} \text{GeV}^{-1}$).

Our model could very well be realized within the context of the pre-big-bang scenario [14], a particular model of inflation inspired by the duality properties of string theory. In the pre-big-bang scenario, perturbations of Kalb-Ramond axions can provide a quasi-scale-invariant (Harrison-Zel'dovich) spectrum of large angular scale cosmic microwave background (CMB) anisotropies [18]. An extension of this mechanism (for massless axions) to the region of the acoustic peaks, showed that a consistency with the current CMB data requires that the internal dimensions contract at a rate faster than the rate at which the external dimensions expand [19]. Therefore, combining our model with the one where the universal axion of string theory triggered the CMB anisotropies, we get further constraints on the extra dimensions of string theory. However, this will be left for future work.

3 Conclusions

We have discussed graviton production in the context of a higher dimensional model, which from a multi-dimensional phase where the extra dimensions are contracting and the external dimensions are expanding, enters into an effectively four-dimensional universe with static extra dimensions. The momenta in the extra space were taken into account. To find an estimate of their contribution to the energy density in the four-dimensional space-time, the momenta in the internal space were integrated out.

The contributions of the internal momenta tend to dominate the gravitational wave spectrum, up to the point of forcing the universe to recollapse at an early stage. However, one can assume that there is an upper cut-off for the internal momenta, given by the maximal frequency of the four-dimensional (external) space-time. The gravitational wave spectrum is constrained by observations. With the assumption that the standard relation between the Hubble parameter and the temperature holds up to the transition time, we derived an upper bound on the temperature at the beginning of the radiation dominated era. Using that Newtonian gravity has been tested successfully down to scales of the order of 1 mm, one obtains a lower bound on the temperature at the transition scale, which depends on the number of extra dimensions. This lower bound on the reheating temperature is, in general, lower than the reheating temperature of standard inflationary models. However, it can be raised if one assumes smaller extra dimensions, which lead to a higher D -dimensional Planck mass. Our final conclusion is that the size R of the internal space must be much larger than Planck length ℓ_{Pl} , namely $R \gg \ell_{\text{Pl}}$.

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